

MU120 RBC



The Open  
University

Mathematics  
and Computing  
A first level  
multidisciplinary  
course

Open **Mathematics**



## **RESOURCE BOOK C**

# *Units 10–13*





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# *Units 10–13*

*Prepared by the course team*



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## Unit 10 Prediction

### Question 1

As an Open University mathematics lecturer, I have a pigeon-hole here at Walton Hall. The pigeon-hole is 10 cm high and I estimate that I get on average 2 cm of mail every week. When I am away on study leave, I ask my secretary to empty it, so that it does not overflow. Just to make sure, she likes to empty it when there is 1 cm of space left. She wants to know how often she will need to empty the pigeon-hole.

- (a) Draw a linear graph to show how the height of the pile of mail,  $h$  cm, changes with time,  $t$  weeks.
- (b) What is the question you want to answer from analysing this graphical linear model?
- (c) What is the answer to your question and what is its interpretation in terms of the original problem?
- (d) Find the gradient of your graph. What is its physical significance?

### Question 2

In this question you are asked to model the same situation as in the previous question but this time with a different choice of variables. Instead of choosing the height of the pile of mail and time, in this question use the amount of space left in the pigeon-hole,  $s$  cm, and time,  $t$  weeks. Initially, at time  $t = 0$  weeks, there is  $s = 10$  cm of space left in the pigeon-hole.

- (a) Draw a linear graph to show how the space left in the pigeon-hole,  $s$  cm, changes with time,  $t$  weeks.
- (b) What is the question you want to answer from analysing this graphical linear model?
- (c) What is the answer to your question and what is its interpretation in terms of the original problem?
- (d) Find the gradient of your graph. What is its physical significance?

### Question 3

Late spring frost is a threat to the blossom in Californian orange groves, so fires are sometimes lit between the trees when a frost is imminent in order to keep the temperature above freezing point. Rather than stay up all night, the workers watch how the temperature is falling after sunset and estimate how low it will go during the night. They then get up and light the fires shortly before their estimate of when the temperature will reach freezing point. From experience they know that the danger of frost occurs on cold clear nights and that the temperature usually drops steadily between sunset and sunrise. These workers are unlikely to write down an algebraic equation but their method of predicting when to light fires is equivalent to an algebraic linear model.

Look at the situation on one particular night. The temperature was  $5^{\circ}\text{C}$  at sunset (6.30 pm),  $4.5^{\circ}\text{C}$  an hour later, and  $4^{\circ}\text{C}$  after a further hour. You

are asked to model the situation to help the workers to decide when to get up and light their fires on this night.

- Define the variables.
- Assume a constant rate of change. Find a numerical value for it.
- Find an initial value.
- Write down the appropriate linear equation.
- What are the limitations of the model?
- How could you use this model to find at what time the temperature will drop to freezing point ( $0^{\circ}\text{C}$ )?
- Find the solution and interpret it in terms of the workers' problem.

### Question 4

Look at the following data, which were obtained by suspending various known masses on an elastic band and measuring the corresponding lengths of the elastic band.

|                  |     |     |     |     |     |     |
|------------------|-----|-----|-----|-----|-----|-----|
| Mass, $x$ (g)    | 100 | 200 | 300 | 400 | 500 | 600 |
| Length, $y$ (mm) | 228 | 236 | 256 | 278 | 285 | 301 |

- Plot the data pairs in the table as points on a graph. Draw 'by eye' the straight line which you think is the 'best fit' to the points you have plotted.
- Use your straight-line graph to estimate:
  - the length of the elastic band before it was stretched—that is, the initial value;
  - the increase in the length of the elastic band caused by the an additional mass of 1 g—that is, the gradient.
- By using your answers to part (b), write down the equation of your 'best fit' line.
- From the equation of your 'best fit' line, estimate the length of the band when it has a mass suspended from it of:
  - 350 g;
  - 800 g;
  - 1200 g.

### Question 5

A slow goods train is scheduled to leave Edinburgh at 10.30 am. An additional holiday train is scheduled to leave half an hour later, and follows the same line to Newcastle (about 125 miles away). The holiday train travels faster than the goods train, and so the timetabling office want to know where to pull the goods train off the main line in order to let the holiday train go through. The goods train averages about 50 mph while the holiday train averages about 90 mph.

- Construct a graphical linear model of the motion of the goods train. On the same diagram construct a graphical linear model of the motion of the holiday train.
- Use your models to predict where the trains will meet if the goods train is not pulled off the main line.
- Interpret your solution in terms of the original problem.



**Question 6**

Suppose you were crossing the English Channel from Folkestone to Calais, a trip of 30 miles. Your ferry left at 20.50 and was scheduled to arrive at 22.40. From the timetable you noticed that another ferry was due to leave Calais at 20.15 and arrive at Folkestone at 22.05, and you were curious as to when you would pass this ferry. Draw graphical linear models to predict when you should look out for the other ferry.

**Question 7**

Look back at Activity 26 of *Unit 10*, involving a graphical model of the supply and demand of soft fruit.

- From your graphs, find linear equations for the demand and supply in terms of the price.
- Use algebraic methods to solve these equations to find the equilibrium price, when the quantity of soft fruit demanded is exactly the same as the quantity supplied.
- How much fruit will be sold at this price according to your model?
- What assumptions were made in this model, and how do these affect the interpretation of the solution?

**Question 8**

Solve the following pairs of simultaneous linear equations for the two unknowns  $x$  and  $y$ , giving your answers rounded to two decimal places.

- $$y = 3 - 2x$$

$$y = 2x - 1$$
- $$y = 10 + 5x$$

$$2y = 3x + 1$$
- $$2y + x = 7$$

$$3y + 2x = 4$$

**Question 9**

For each of the following sets of conditions, draw a graph and mark on it the region where all of the conditions hold true.

- $x \geq 0$ ,  $y \geq 0$ ,  $2x + 3y \leq 60$ .
- $x + y \leq 2$ ,  $x + y \geq -2$ ,  $x - y \geq -2$ ,  $x - y \leq 2$ .
- $x \leq 50$ ,  $x/6 + y/8 \leq 10$ ,  $y + 4x \geq 80$ ,  $2y + x \geq 60$ .



**Question 10**

- (a) Sketch the feasible region determined by the following set of inequalities and work out the coordinates of the vertices of the region.

$$x \geq 0, \quad y \geq 0, \quad 2x + y \leq 8, \quad x + 3y \leq 9.$$

- (b) Find graphically the point of the feasible region in (a) which maximizes  $P$  in the cases:

(i)  $P = x + y$ ;

(ii)  $P = 3x + y$ .

**Question 11**

- (a) Sketch the feasible region determined by the following set of inequalities and work out the coordinates of the vertices of the region.

$$x \geq 0, \quad y \geq 0, \quad x + 2y \leq 12, \quad 5x + y \leq 15, \quad 4x + 3y \leq 30.$$

- (b) Find graphically the point of the feasible region in (a) which maximizes  $P$  in the cases:

(i)  $P = x + y$ ;

(ii)  $P = 7x + 2y$ .

**Question 12**

Suppose that you have agreed to bake some cakes for a party at a club you belong to. The club will provide all the necessary ingredients and a fully equipped kitchen, and will also pay you 80p each for fruit cakes and £1 each for iced cherry cakes. Suppose you estimate that fruit cakes take one hour of your time to prepare, and  $1\frac{1}{2}$  hours in the oven; and that iced cakes take two hours of your time but only use one hour of oven time. You have 18 hours of time available for preparation, and 14 hours of time to use the oven (the oven is only big enough for one cake at a time).

- (a) Model this problem as a linear programming problem. Sketch the feasible region, and hence determine how many cakes of each type you should make in order to maximize the income you will receive.
- (b) Suppose the price you receive for iced cakes increases to £1.50 each. Does this change your answer to (a)?
- (c) What would happen if the price for iced cakes rose to £1.70 each?

### Question 13

Suppose that you have been reading about a new wonder diet in which you eat only bread and low-fat cheese. The nutritional details for a slimmers' bread and a low-fat cheese are given in the following table.

|                    | Bread | Cheese | Required daily intake (g) |
|--------------------|-------|--------|---------------------------|
| Protein (%)        | 8     | 16     | 75                        |
| Carbohydrates (%)  | 46    | 6      | 250                       |
| Fat (%)            | 2     | 8      | 70                        |
| Calories per 100 g | 235   | 166    |                           |

The table also shows the required minimum daily intake of protein, carbohydrates and fat for a healthy diet. (Of course, a healthy diet also requires vitamins, minerals, and so on—but these are ignored here or you would not be able to solve the problem without using a computer.) How much bread and cheese should you eat each day so that your daily diet is healthy as far as protein, carbohydrates and fat are concerned, while at the same time minimizing your daily calorie intake?

- Model this problem as a linear programming problem, using as variables your daily intake in 100 g units of bread ( $x$ ) and cheese ( $y$ ), and draw the feasible region.
- What can you say about the protein requirement?
- Use your diagram to find the solution to your problem graphically.
- Find the optimal solution more accurately using algebra to solve the appropriate simultaneous equations.
- How many calories per day will you be getting?



# Unit 11 Movement

## Question 1

- (a) Write down the equation of a parabola with its axis parallel to the  $y$ -axis and with its vertex at:
- (i)  $(0, 4)$ ;      (ii)  $(4, 0)$ ;      (iii)  $(1, -2)$ .
- (b) Write down the equations of two parabolas with axes parallel to the  $y$ -axis and with vertex at  $(6, -4)$ .

## Question 2

Write down the equation of the parabola with axis parallel to the  $y$ -axis, vertex at  $(-1, 2)$ , and which goes through the point  $(1, 14)$ .

## Question 3

Write down the coordinates of the vertex of each of the following parabolas.

- (a)  $y = (x + 1)^2 + 1$   
(b)  $y = x^2 + 1$   
(c)  $y = (x - 2)^2 + 1$   
(d)  $y = 2(x - 3)^2 + 2$

## Question 4

Suppose that a marine biologist is researching on certain type of (very fast growing) coral. An experiment is performed on a piece of coral by weighing it every month. The data are collected in the table below.

|               |      |      |      |      |      |      |      |      |      |      |       |
|---------------|------|------|------|------|------|------|------|------|------|------|-------|
| Time (months) | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11    |
| Weight        | 1580 | 2040 | 2880 | 3330 | 4590 | 6210 | 6430 | 8400 | 9930 | 9960 | 12380 |

- (a) Enter the data into your calculator and draw a scatterplot of the data. Do you think that it is reasonable to fit a straight line to the data?
- (b) Use the linear regression features of your calculator to fit a straight line to the data. Use the regression line to predict what the coral will weigh in month 20.
- (c) The marine biologist has a theory that the growth of the coral is purely at the surface and so is proportional to the surface area of the coral. This suggests that a quadratic model is more appropriate for the data. Use your calculator to fit a quadratic model to the data, and use it to predict the weight of the coral in month 20.
- (d) Comment on any differences between your predictions for the weight of the coral at 20 months.



### Question 5

Suppose that an experimenter wants to analyse the motion of a stone as it is thrown from a cliff. The experimenter starts a watch and then throws the stone from the cliff. The experiment is filmed and the data from the experiment are presented as the graph in Figure 1.

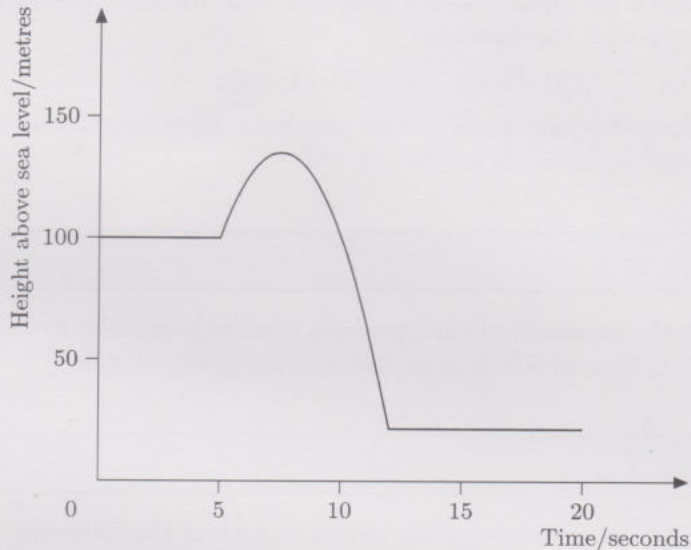


Figure 1

- What is the height above sea level of the top of the cliff?
- How long after starting his watch did the experimenter throw the stone?
- When did the stone return to a height level with the cliff edge?
- At what time did the stone reach its maximum height? How high did the stone go?
- Using the answers to the previous three parts of the question to define three points on the parabola, use your calculator to find the equation of the best fit parabola which goes through these three points.
- How high above sea level is the bottom of the cliff?
- Use your calculator to predict how fast the stone was travelling when it hit the ground (that is, find the gradient of the parabola when the stone hit the ground).

### Question 6

A good model of certain towns and villages is a roughly circular shape. Sometimes the population density along a diameter can be modelled as a parabola. The population density along the diameter of a town modelled in this way is shown in Figure 2.

- Calculate the equation of the parabola that the graph represents (either algebraically or by fitting a parabola using your calculator).
- Use your equation to predict the population density 200 m from the centre of the village.

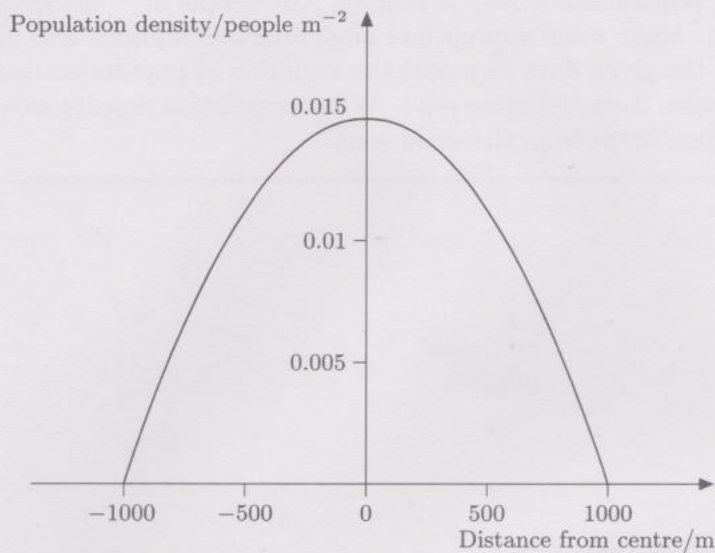


Figure 2

### Question 7

Although many towns are roughly circular in shape, some villages have evolved along a main street in such a way that their population density has its peak value alongside the road and falls away each side of the road, as illustrated by Figure 3.

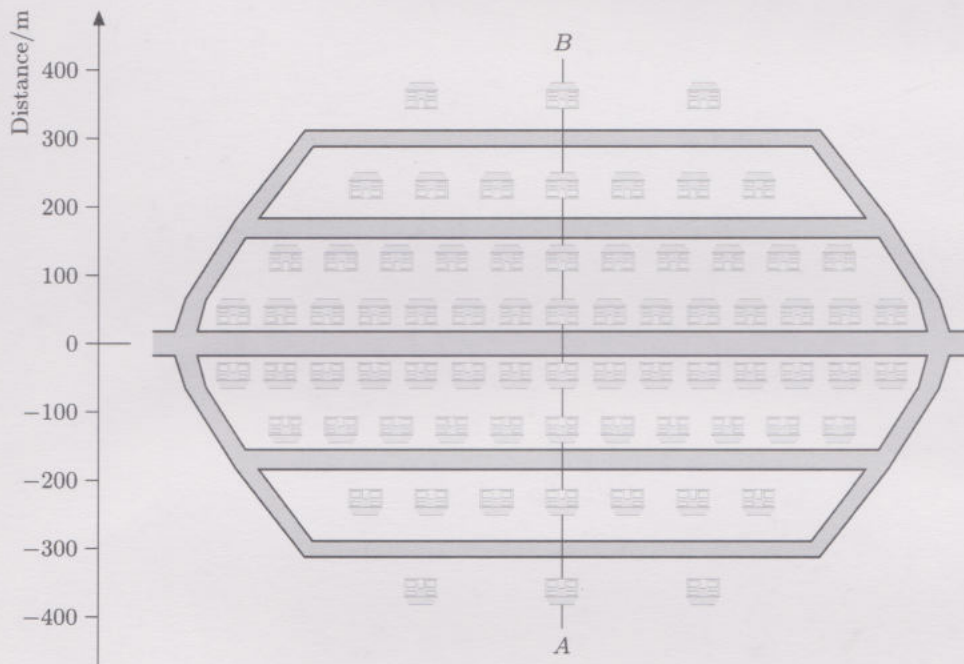


Figure 3

- (a) Sketch how you think the population density is likely to vary along the line  $AB$  in Figure 3. What curve could model this variation? What would you have to assume about the population density outside the village boundary?

- (b) The population density is roughly  $0.01 \text{ people m}^{-2}$  alongside the main road. Make some appropriate modelling assumptions and use them and the given data to model the variation of population density with distance from the main road. What population density would you predict 200 m from the main road?
-



## Unit 12 Growth and decay

### Question 1

For the following examples, calculate the population for the first four generations by direct calculation. Then devise a formula for the  $n$ th generation and use the formula to calculate the population in the 20th generation.

- (a) A population of insects that starts with 5 members and triples every month.
- (b) A fungus which adds 1% to its mass every day and starts off weighing 23.1 grams.
- (c) A person who has just inherited £1000 and spends 10% of the remaining money each month.

### Question 2

For the examples in Question 1, calculate the sum of the populations in the first four generations (your answer to Question 1 should help). Use the general formula for the sum of the first  $n$  generations to write down the sum of the first  $n$  generations for each of the examples in the previous question. Use the formula to calculate the sum of the first 20 generations in each case.

### Question 3

Solve the following equations for the unknown  $x$ , giving your answers correct to two decimal places.

- (a)  $20 = 10^x$
- (b)  $0.25 = 1/\log_{10} x$
- (c)  $2\log_{10}(x^3) = 6$
- (d)  $2^x = 100$

Hint: in (a) and (d), take  $\log_{10}$  of both sides of the equation.

### Question 4

In Question 1, the case of a person reducing a £1000 inheritance by spending 10% of the remaining money each month was considered. The formula for the amount of money left in the  $n$ th month was found to be

$$£1000 \times 0.9^{n-1}.$$

In this question, you are asked to try to calculate when the person spends the last penny (£0.01) of the inheritance. (The answer will only be approximate, because the effect of rounding to the nearest penny is neglected at intermediate steps of the calculation—that is, only whole numbers of pence can be withdrawn each month.)

- (a) Calculate approximately, by trial and error, in which month the person will spend the last penny of the inheritance.

Hint: evaluate the given formula for various values of  $n$ .

- (b) Now use logarithms to solve the problem.

### Question 5

For the following functions, calculate the population doubling times.

- (a)  $y = 209 \times 3^x$
- (b)  $y = 0.43 \times (1.2)^x$
- (c)  $y = 65 \times (0.9)^x$

### Question 6

Write down the equation of the exponential model in the form  $y = a \times b^x$  for each of the following cases.

- (a) Initial population 76, population doubling time 2 minutes.
- (b) Initial population 10, population doubling time 6 minutes.
- (c) Initial population 1000, half-life 23 minutes.

### Question 7

The radioactive isotope uranium 239 has a half-life of 23.5 minutes.

- (a) What is the 'quarter-life' of the isotope—that is, the time until the number of atoms of the isotope has decreased by a factor of four.
- (b) Calculate the 'third-life' of the isotope—that is, the time until the number of atoms of the isotope has decreased by a factor of three.

### Question 8

An environmental pressure group claims that the number of individuals in a certain species of bird is declining exponentially. The data that the group gives to support its claim is from a project of annual censuses conducted over ten years. The data are collected together in the following table (the years are numbered from the start of the project; 1985 is year 0).

|             |     |     |     |     |     |     |     |     |     |     |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Year number | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
| Population  | 515 | 481 | 441 | 390 | 339 | 307 | 275 | 227 | 195 | 164 |

- (a) Fit an exponential model to the data as the pressure group suggests. Use your model to predict the population of birds in the year 2000 (year 15 in the numbering system of the project).
- (b) Now assume that the decline is linear and fit a linear model to the data. Use your linear model to make another prediction of the population in the year 2000.
- (c) Comment on your predictions for the population in the year 2000.

### Question 9

In this question you are to investigate the statement 'exponential growth increases much faster than any polynomial'.

When mathematicians make statements like this they are usually referring to what happens 'eventually' as numbers get larger and larger. So, if the above statement is true, it must 'eventually' be true for even the slowest



exponential growth. With this in mind, the first two parts of this question ask you to find the slowest exponential growth that you can calculate with on your calculator—that is, the one with the growth factor that is as close to 1 as possible.

- (a) To find the closest number to 1 that your calculator can calculate with, try evaluating the following sequence of expressions of the form  $b - 1$  for increasingly smaller values of  $b$ :

$$\begin{aligned} 1.000\,01 - 1 \\ 1.000\,001 - 1 \\ 1.000\,0001 - 1 \\ \vdots \end{aligned}$$

At what point do you notice a difference in the behaviour of your calculator? What is the smallest number  $b$  for which your calculator gives the correct answer?

- (b) Your answer to part (a) means that the slowest growing exponential that your calculator can compute is

$$y = b^x$$

where  $b$  is the smallest value of  $b$  that you obtained in part (a). What value of  $x$  is needed for  $y = b^x$  to be bigger than 10 for this value of  $b$ ? Give your answer to three significant figures.

Hint: write down  $\log_{10}$  of both sides of the equation.

- (c) Now pick a rapidly increasing polynomial and see at what point this slowest growing exponential model crosses and overtakes it. For example, consider the following polynomial:

$$y = 1000x^2 + 1000$$

How large does  $x$  have to be before the slowest growing exponential model is bigger than this (only an approximate answer to one significant figure is required)?

Hint: enter the two functions into your calculator and produce a table of values; use your answer to the previous part of the question as a starting point for your table.

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## Unit 13 Baker's dozen

### Question 1

Suppose that it has taken you one hour to read the first twenty pages of a book and assume that you will read the rest at the same speed.

- (a) Set up a mathematical model to help you predict how long it will take you to read any given number of pages of the book, by:
  - (i) choosing two appropriate variables;
  - (ii) making an assumption which leads to a proportionality relationship;
  - (iii) writing down this relationship, using the relevant constant of proportionality.
- (b) Use the model to estimate how long it will take you to read:
  - (i) the first chapter of 115 pages;
  - (ii) the third chapter of 46 pages;
  - (iii) the whole book of 670 pages.

### Question 2

Suppose that you were preparing a meal for some friends and were making a main course from a recipe which required a 15 cm diameter serving dish. At the last minute, when you had put all the ingredients together according to the recipe, you realized that your round serving dish had a diameter of 20 cm. How would the depth of the food in the serving dish be affected by using your dish instead of that given in the recipe (without increasing the ingredients)?

### Question 3

Your recipe book lists the sizes for round cake tins but you have only got square cake tins. How do you convert from round to square (assuming you want the cakes to be the same height)? A particular recipe asks for an 8-inch diameter round cake tin. What is the corresponding length of side for a square tin of the same height?

Hint: the formula for the area of a square with side  $x$  inches is  $x^2$  square inches; the formula for the area of a circle of radius  $r$  inches is  $\pi r^2$  square inches.

### Question 4

You are swapping recipes with a friend and find that you can never make your friend's recipes work. Eventually you trace the problem to the different power ratings of your microwave ovens. You have a 650-watt microwave while your friend has a 500-watt microwave.

- (a) State an assumption which leads to a simple relationship between microwave oven power and cooking times. What is this relationship?

Hint: if you have difficulty coming up with a relationship between power and time then you might like to think about the definition of the watt: one watt is one joule (a unit of energy) per second.

- (b) Make an estimate of the range of validity of your model.
- (c) In order to prevent confusion in the future, you decide to produce a table which lists the times for both microwaves for some items that you cook regularly. Use your relationship from part (a) to fill in the table, rounding your answers to the nearest half minute.

| Food item           | Cooking time (minutes) |               |
|---------------------|------------------------|---------------|
|                     | 500-watt oven          | 650-watt oven |
| One corn-on-the-cob |                        | 5             |
| 225 g oven chips    | 9                      |               |
| Whole kipper        |                        | 6             |
| 4×225 g cod steaks  |                        | 12            |
| 2 kg beef           | 64                     |               |
| 2×100 g pork chops  | 6                      |               |

### Question 5

The size of televisions is often quoted as the length of the diagonal of the screen. However, people often subconsciously judge the size of the screen by the screen area.

- (a) How much bigger would someone perceive a 24-inch television compared with a 21-inch television? (Assume that both television screens are the standard shape with sides in the ratio 4 : 3.)
- (b) Generalize your answer of the previous part from a 24-inch television to an  $x$ -inch television. How much bigger is an  $x$ -inch television perceived compared with a 21-inch model?

### Question 6

The volume of a sphere of radius  $r$  is  $\frac{4}{3}\pi r^3$  and the surface area is  $4\pi r^2$ . Suppose that you are a manufacturer of spherical sweets and you already make sweets 1 cm in diameter. Some market research has indicated that there would be a market for a giant version of these sweets with a diameter of 3 cm.

- (a) By what factor would you scale up the ingredients for the coating and the interior of one sweet?
- (b) There are currently 600 small sweets to a packet. How many giant sweets would need to be put into a packet of about the same weight?
- (c) Use your answers to the previous two parts of this question to estimate by how much the ingredients for the interiors and the coatings need to be scaled for one packet of giant sweets.
- (d) The sweets are actually manufactured in batches of 100 packets. What would the scaling of the ingredients be for each batch of the giant sweets?



### Question 7

A DIY enthusiast is trying to estimate how long it would take to dig a swimming pool in the garden. It takes 2 hours to dig a test hole which measures 1 m by 1 m by 0.5 m.

- Devise a formula for how long it will take to dig a hole which measures  $x$  m by  $x$  m by  $x/2$  m.
- How long will it take to dig the whole swimming pool, which measures 5 m by 5 m by 2.5 m?

### Question 8

One method that astronomers use to find the distance from the Earth to a star is to use a relationship between the apparent brightness and the distance from the Earth. The apparent brightness of a star viewed from the Earth is determined by the intensity of light reaching the Earth. Astronomers have instruments for measuring the intensity of light from stars.

Now consider a star which emits a total light energy of  $E$  every second. Assume that the light from the star radiates equally in all directions. So when the light has travelled a distance  $r$  the light will be spread over the surface of an imaginary sphere as in Figure 4.

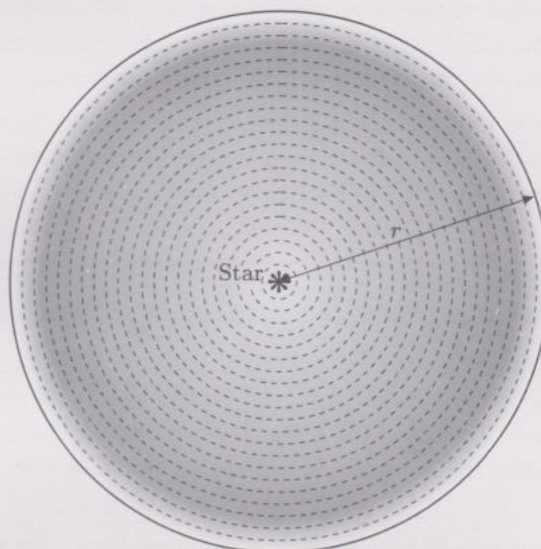


Figure 4

Let the intensity of the light at any point of unit area on the surface of this imaginary sphere be  $I$ .

- Calculate the total light energy per second passing the surface of the imaginary sphere in terms of  $I$  and  $r$ . (The formula for the surface area of a sphere of radius  $r$  is  $4\pi r^2$ .)
- Since energy must be conserved, the energy you calculated in the previous part must equal  $E$ , the total light energy emitted by the star per second. Hence derive a formula for how the intensity of light  $I$  varies with the distance from the star  $r$ .

*Intensity of light* is defined to be the amount of light energy hitting a unit area per second.



### Question 9

It is useful to be able to recognize the graphs of the common functions. Practise your skill now by trying to match a function with its graph. The six types of functions that you have to choose from are the following.

- 1 A sinusoidal function of the form  $y = A \sin x$ .
- 2 An exponentially decreasing function of the form  $y = ab^x$  with  $b < 1$ .
- 3 An exponentially increasing function of the form  $y = ab^x$  with  $b > 1$ .
- 4 A quadratic function of the form  $y = a(x - k)^2 + l$ .
- 5 A linear function of the form  $y = mx + c$ .
- 6 A inversely proportional function of the form  $y = k/x$ .

The graphs of these functions are shown in Figure 5.

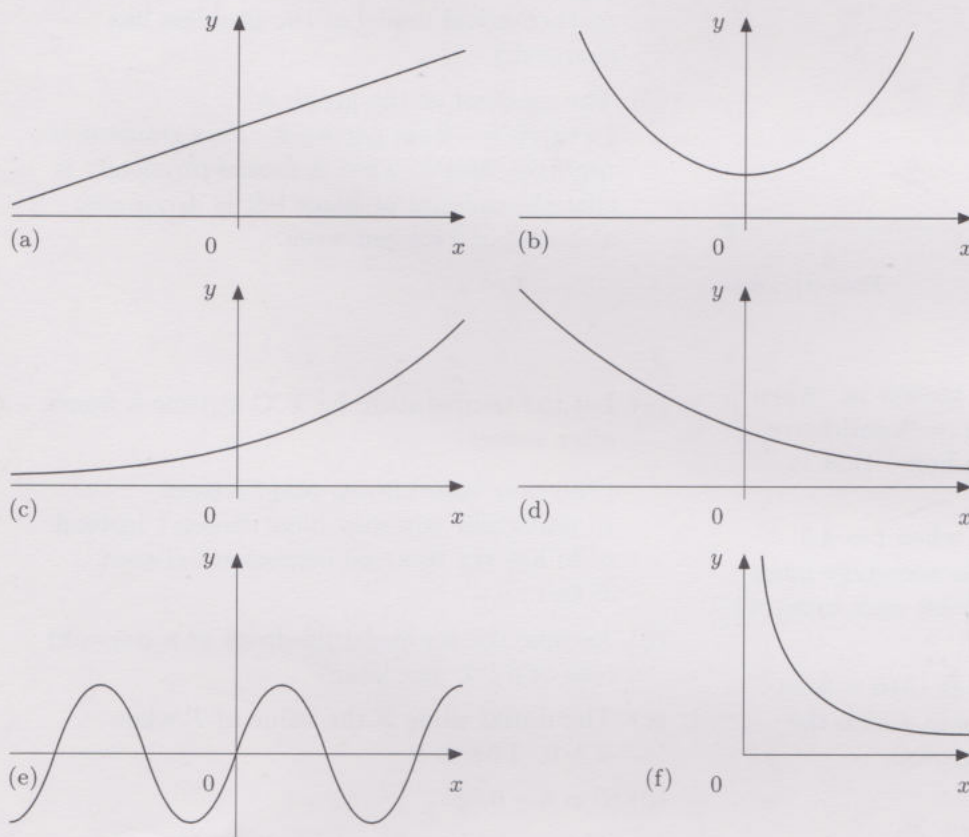


Figure 5

Write down your answer as a sequence of pairs: for example, if you think that the sinusoidal function is represented by graph (b) then write 1b.

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# Solutions

## Unit 10

1

- (a) Figure 6 shows an appropriate linear graph.

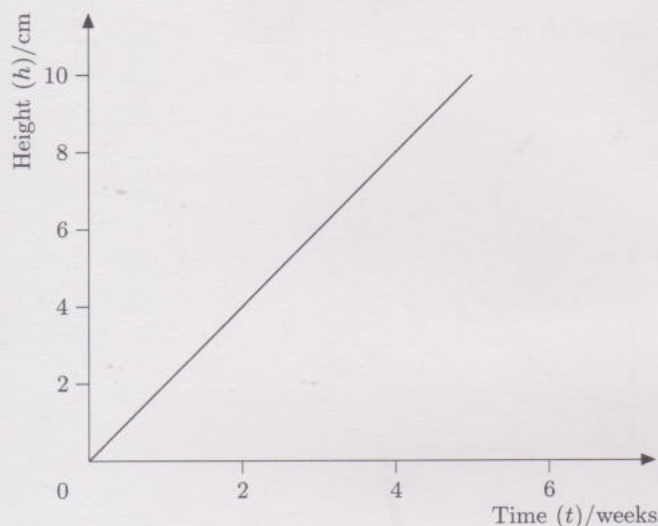


Figure 6

- (b) The question you want to answer is: 'when does the pile of mail reach one centimetre from the top of the pigeon-hole—that is, when does  $h = 9$  cm?'
- (c) From the graph,  $h = 9$  cm when  $t = 4.5$  weeks. This means that the secretary must empty the pigeon-hole at least once every  $4\frac{1}{2}$  weeks.
- (d) The gradient of the graph is  $10/5 = 2$  cm per week, which is the rate at which the height of the pile of mail grows.

2

- (a) Figure 7 shows an appropriate linear graph.

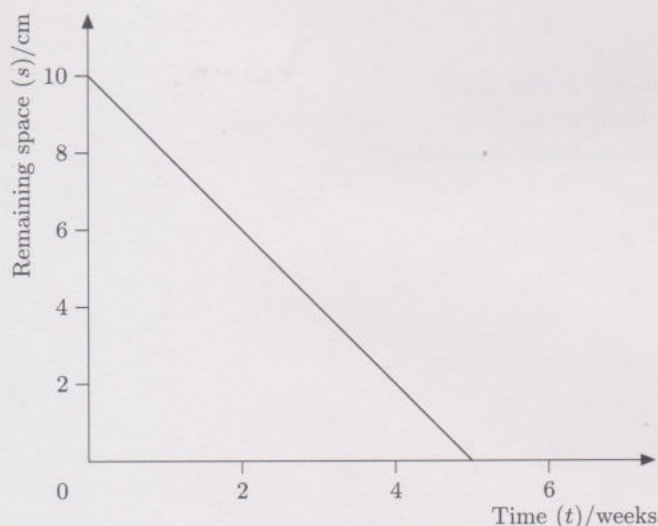


Figure 7

- (b) The question you want to answer is: 'when is the space left in the pigeon-hole equal to one centimetre—that is, when is  $s = 1$  cm?'
- (c) From the graph,  $s = 1$  cm when  $t = 4.5$  weeks. This means that the secretary must empty the pigeon-hole at least once every  $4\frac{1}{2}$  weeks.
- (This is the same result as the previous question. This *must* happen because no aspect of the real problem changed; only our mathematical model of the problem has changed.)
- (d) The gradient of the graph is  $(-10)/5 = -2$  cm per week. This gradient is negative, and so what it means physically is that the amount of space left is *decreasing* at a rate of 2 cm per week.

3

- (a) Let the temperature be  $T$  °C at time  $h$  hours after sunset.
- (You may have chosen other letters. In particular you may have chosen  $t$  instead of  $h$ ; but try to avoid confusion between  $T$  and  $t$ .)
- (b) Assume the temperature drops at a constant rate of  $0.5$  °C per hour.
- (c) The initial value is the value of  $T$  when  $h = 0$ . This is 5.
- (d)  $T = 5 - 0.5h$ .
- (e) The model is only valid between sunset ( $h = 0$ ) and sunrise (probably around  $h = 12$ ).
- (f) The temperature has dropped to freezing point when  $T = 0$ . So you need to use the equation to find the value of  $h$  when  $T = 0$ .
- (g) Substitute  $T = 0$  into the equation:

$$\begin{aligned} 0 &= 5 - 0.5h \\ 0.5h &= 5 \\ h &= 10 \end{aligned}$$

The temperature will fall to  $0$  °C ten hours after sunset, that is at 4.30 am. The workers would, however, need to light fires *before* this. They would probably allow a good safety margin, and get up and light the fires between 3.30 and 4.00 am.



4

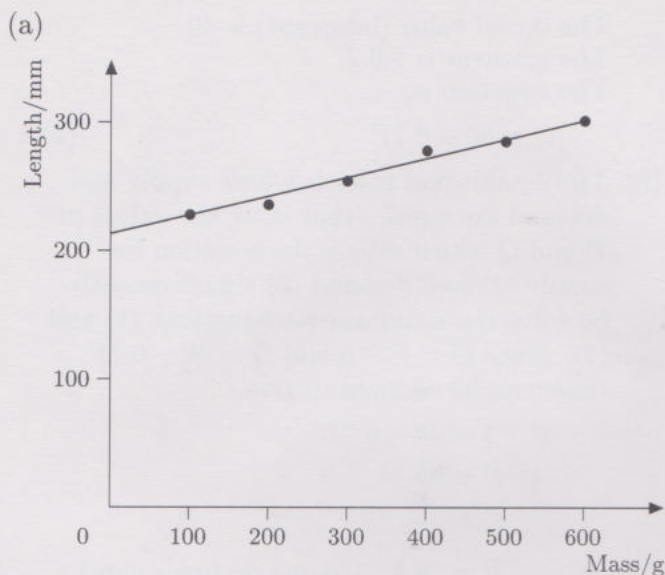


Figure 8

- (b) (i) The intercept of the graph is at 213 mm (your value should be something similar, say plus or minus 5): this is the length of the elastic band before it was stretched.

(ii) The increase in length per additional 1 g mass is about 0.15 mm. To calculate this, choose two points on your 'best fit' line which are a reasonable distance apart and calculate the change in  $y$  divided by the change in  $x$ : for example, if the points (100, 228) and (600, 301) are on your line, then the gradient is:

$$\frac{301 - 228}{600 - 100} = \frac{73}{500} = 0.15$$

(to two decimal places)

- (c) The equation of the 'best fit' line for the values found in part (b) is:

$$y = 0.15x + 213$$

(You may have slightly different values.)

- (d) The following are the values obtained from the equation found in part (c). Your values may be slightly different.

(i)  $0.15 \times 350 + 213 = 267$

So the length is 267 mm for a 350 g mass.

(ii)  $0.15 \times 800 + 213 = 333$

So the length is 333 mm for a 800 g mass.

(iii)  $0.15 \times 1200 + 213 = 393$

So the length is 393 mm for a 1200 g mass.

Of course, all of these values assume that the linear model for expansion is valid. This is probably true for the 350 g mass, since this is inside the range of the given data (interpolation). The other two masses, however, are outside the given data range (extrapolation), and so the model may not

be valid for these values. In fact, for large masses over a certain mass (known as the elastic limit), the linear model fails (for example, in this case the elastic band snapped when the 1200 g mass was attached).

5

- (a) Define the variables as distance  $d$  miles from Edinburgh and time  $t$  hours after 10.30 am.

Assuming that the goods train is travelling at a constant speed of 50 mph starting at  $d = 0$  and  $t = 0$ , its motion can be modelled by a straight line of gradient 50 passing through the origin (0, 0). The model is valid until either the train reaches Newcastle or it is pulled off the main line.

Assuming that the holiday train is travelling at a constant speed of 90 mph, then its motion can also be modelled by a straight line. However, the holiday train sets off half an hour later, so it starts when  $t = 0.5$  and  $d = 0$ . Its motion is therefore modelled by a straight line passing through the point (0.5, 0) and with a gradient of 90. The model is valid until the train reaches Newcastle.

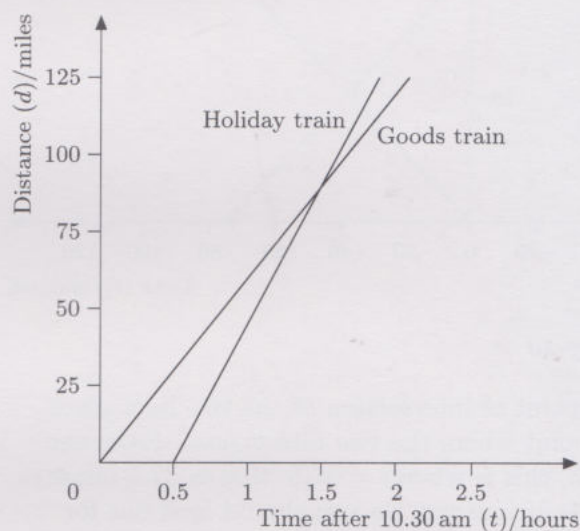


Figure 9

- (b) The model predicts that the trains will meet after about 1.1 hours, 56 miles from Edinburgh.
- (c) No allowance was made for variation in speed or for the time needed to re-route the goods train. So, a good safety margin is needed. The goods train must be pulled off the main line well before 56 miles, preferably at a suitable place about 50 miles from Edinburgh.



**6** Let  $d$  be the distance from Folkestone measured in miles and let  $t$  be the time after 20.50 measured in minutes. (You could equally well have chosen to measure the time in minutes after 20.15.) Both ferry journeys can be modelled by linear models if you assume that each ferry travels at a constant speed. Each model will be valid for times less than or equal to the 110-minute journey time (and also for distances up to 30 miles).

For your ferry,  $d = 0$  when  $t = 0$  and  $d = 30$  when  $t = 110$ . So your ferry's journey is represented by the straight line through the two points  $(0, 0)$  and  $(30, 110)$ .

For the other ferry, the timetable says that the ferry was in Calais harbour at 20.15; so  $d = 30$  when  $t = -35$ . It is due to arrive at Folkestone at 22.05; so  $d = 0$  when  $t = 75$ . So the other ferry's journey is represented by the straight line through the two points  $(-35, 30)$  and  $(0, 75)$ .

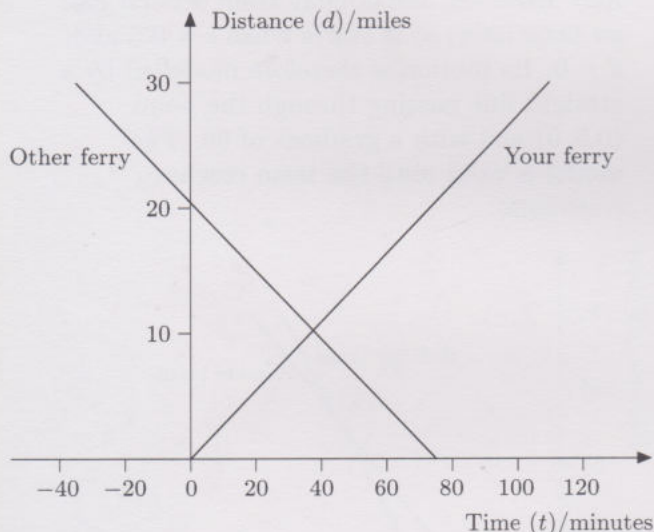


Figure 10

The point of intersection of the two lines gives the point where the two ferries pass. From the graph, this is when  $t = 37.5$ , that is 37.5 minutes into your journey. So you should look out for the other ferry from about 21.25 onwards.

**7**

- (a) Looking at the models for supply and demand separately gives the following.

*Supply*

The initial value (intercept) is  $-5$ .

The gradient is  $1$ .

The equation is:

$$Q = P - 5 \quad (1)$$

*Demand*

The initial value (intercept) is  $40$ .

The gradient is  $-0.7$ .

The equation is:

$$Q = 40 - 0.7P \quad (2)$$

- (b) The equilibrium price is where supply and demand are equal—that is, at the values of  $P$  and  $Q$  which satisfy the equation for supply (1) and demand (2) simultaneously. So solve the simultaneous equations (1) and (2). Since  $Q = P - 5$  and  $Q = 40 - 0.7P$ , these can be equated to give:

$$P - 5 = 40 - 0.7P$$

$$1.7P = 45$$

$$P = \frac{45}{1.7}$$

$$P = 26.5 \quad (\text{to one decimal place})$$

So the equilibrium price is around 26 to 27p.

- (c) Equation (1) gives  $Q = 21.5$  (to one decimal place). Checking in equation (2) also gives  $Q = 21.5$  (to one decimal place). So about 21.5 million kg of soft fruit will be sold.
- (d) Linear models have been assumed; these do not take into account any change in such factors as inflation, bad weather, seasons. Hence the predictions of the equilibrium price and of the corresponding amount of soft fruit sold are only approximate.

**8**

$$(a) \quad y = 3 - 2x \quad (3)$$

$$y = 2x - 1 \quad (4)$$

Substitute for  $y$  in equation (4) using equation (3) and simplify:

$$3 - 2x = 2x - 1$$

$$4 = 4x$$

$$1 = x$$

Substitute this into equation (3) to get:

$$y = 3 - 2 \times 1 = 1$$

So the solution is  $x = 1$  and  $y = 1$ .

Checking in equation (4) gives  $1 = 2 \times 1 - 1$ , which is correct.

$$(b) \quad y = 10 + 5x \quad (5)$$

$$2y = 3x + 1 \quad (6)$$

Substitute for  $y$  in equation (6) using equation (5) and simplify:

$$2 \times (10 + 5x) = 3x + 1$$

$$20 + 10x = 3x + 1$$

$$7x = -19$$

$$x = -\frac{19}{7}$$



Substitute this into equation (5) to get:

$$y = 10 + 5 \times \left(-\frac{19}{7}\right) = -\frac{25}{7}$$

So the solution is  $x = -2.71$  and  $y = -3.57$  (rounded to two decimal places).

Checking in equation (6) gives

$$2 \times \left(-\frac{25}{7}\right) = 3 \times \left(-\frac{19}{7}\right) + 1, \text{ which is correct.}$$

$$(c) \quad 2y + x = 7 \quad (7)$$

$$3y + 2x = 4 \quad (8)$$

Rearrange (7) to make  $y$  the subject:

$$2y = 7 - x$$

$$y = \frac{7}{2} - \frac{x}{2}$$

Substitute this into equation (8) and simplify:

$$3 \times \left(\frac{7}{2} - \frac{x}{2}\right) + 2x = 4$$

$$\frac{3 \times 7}{2} - \frac{3 \times x}{2} + 2x = 4$$

$$10.5 - 1.5x + 2x = 4$$

$$0.5x = -6.5$$

$$x = -13$$

Substitute this into equation (7) and solve for  $y$ :

$$2y + (-13) = 7$$

$$2y = 20$$

$$y = 10$$

So the solution is  $x = -13$  and  $y = 10$ .

Checking in equation (8) gives

$$3 \times 10 + 2 \times (-13) = 4, \text{ which is correct.}$$

9

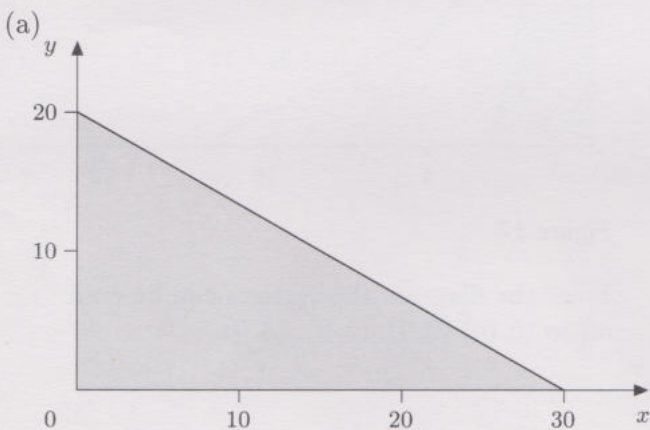


Figure 11

(b)

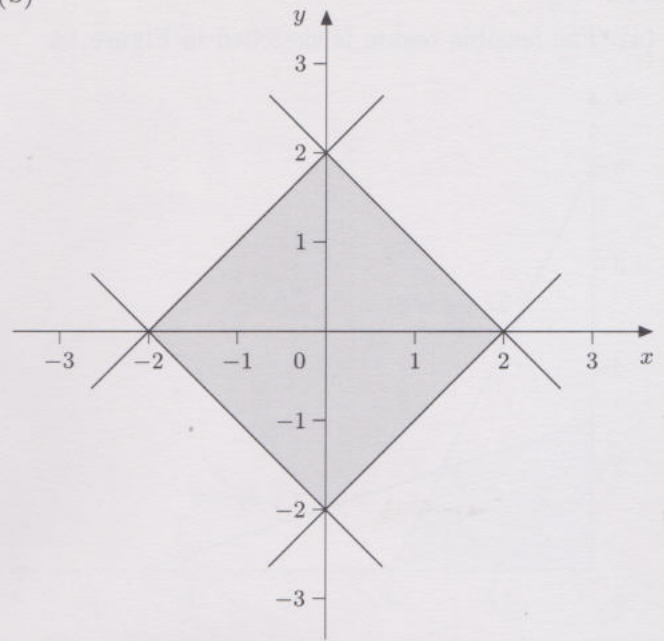


Figure 12

(c)

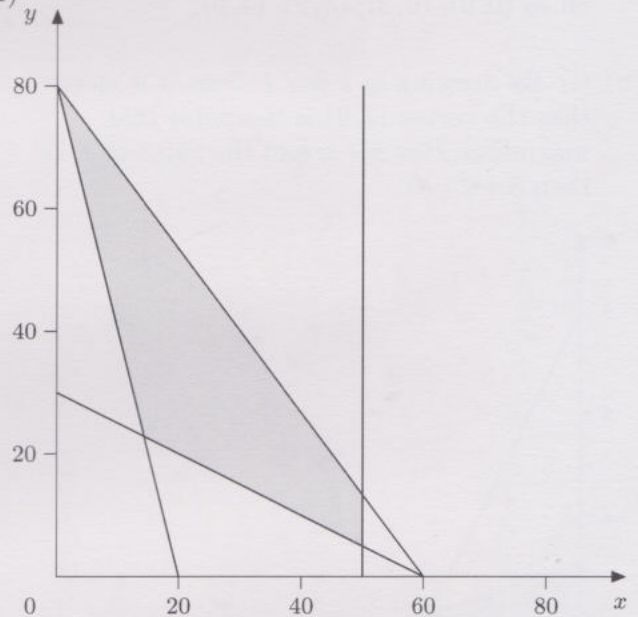


Figure 13

10

(a) The feasible region is sketched in Figure 14.

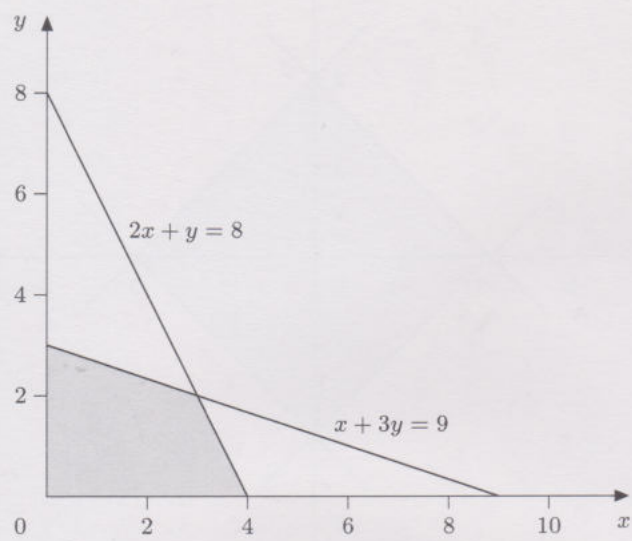


Figure 14

From the diagram the vertices can be read off as (0,0), (0,3), (3,2), (4,0).

(b) (i) By drawing in a few  $P$ -lines it is clear that the vertex (3,2) is the point that maximizes  $P = x + y$  and the value of  $P$  is then  $3 + 2 = 5$ .

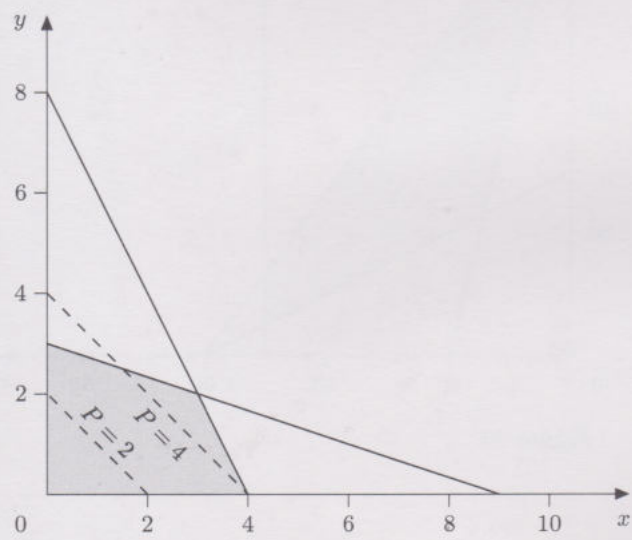


Figure 15

(ii) By drawing in a few  $P$ -lines it is clear that the vertex (4,0) maximizes  $P = 3x + y$  and the value of  $P$  is then  $3 \times 4 + 0 = 12$ .

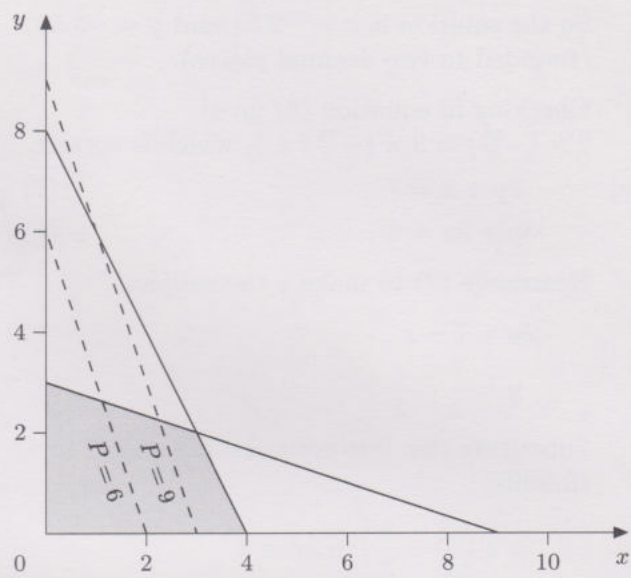


Figure 16

11

(a) The feasible region is sketched in Figure 17.

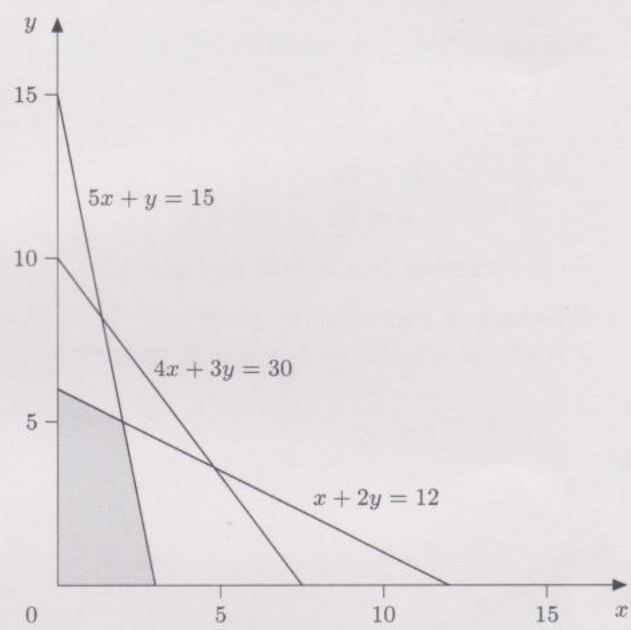


Figure 17

From the diagram the vertices can be read off as (0,0), (0,6), (2,5), (3,0).



- (b) (i) By drawing in a few  $P$ -lines it is clear that the vertex  $(2, 5)$  maximizes  $P = x + y$  and the value of  $P$  is then  $2 + 5 = 7$ .

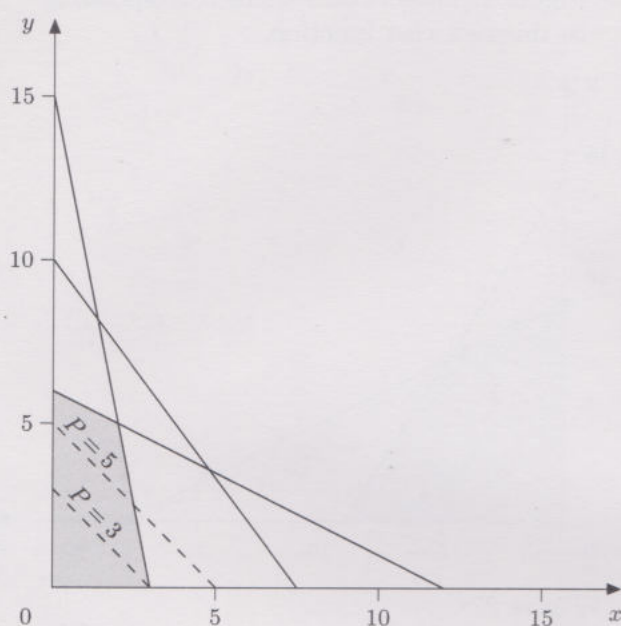


Figure 18

- (ii) By drawing in a few  $P$ -lines it is clear that the vertex  $(2, 5)$  maximizes  $P = 7x + 2y$  and the value of  $P$  is then  $7 \times 2 + 2 \times 5 = 24$ .

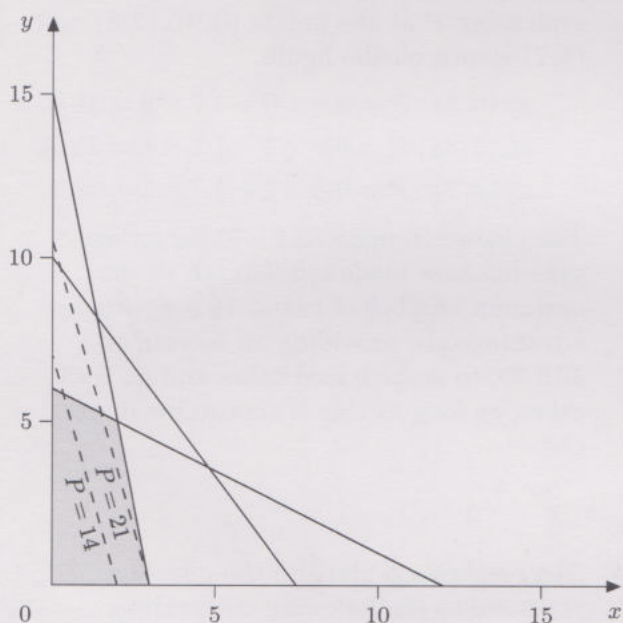


Figure 19

## 12

- (a) Start by defining variables for the problem. Let  $x$  be the number of fruit cakes made and let  $y$  be the number of iced cakes made.

The information given in the question can be summarized by the following table.

|                             | Fruit cakes<br>$x$ | Iced cakes<br>$y$ | Total<br>available |
|-----------------------------|--------------------|-------------------|--------------------|
| Preparation<br>time (hours) | 1                  | 2                 | 18                 |
| Oven time<br>(hours)        | 1.5                | 1                 | 14                 |
| Price (£)                   | 0.80               | 1.00              |                    |

The restriction that the number of preparation hours is limited to 18 translates to the following inequality:

$$x + 2y \leq 18$$

Similarly, the restriction that the number of oven hours is limited to 14 translates to the following inequality:

$$1.5x + y \leq 14$$

Further constraints are implicit in the question. Obviously the number of each type of cake must be positive, so that  $x \geq 0$  and  $y \geq 0$ . Also only whole numbers of cakes can be made.

The function to be maximized is the income:

$$P = 0.8x + y$$

The feasible region and two  $P$ -lines are shown in Figure 20.

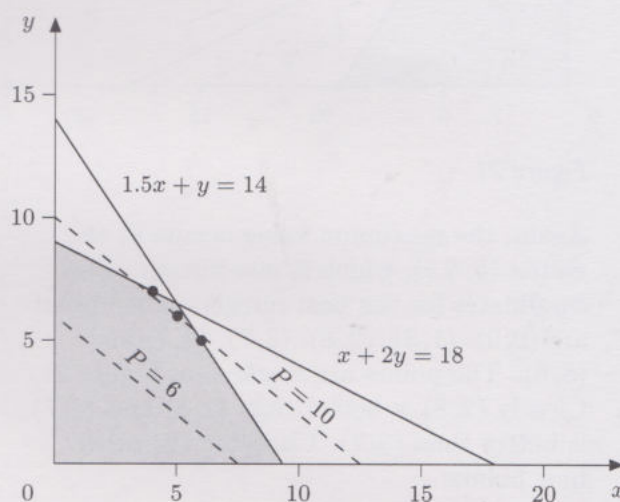


Figure 20

So, clearly, the maximum value occurs at the vertex where the lines  $x + 2y = 18$  and  $1.5x + y = 14$  intersect. By solving this pair of simultaneous equations or by examining the graph, the coordinates of this vertex are found to be  $(5, 6.5)$ ; and the  $y$ -coordinate is non-integer.

To find the optimal integer-valued solution, examine the integer-valued points near this vertex. Bearing in mind the  $P = 10$  line, the

candidates are the points (4, 7), (5, 6) and (6, 5) marked on Figure 20. From the figure, it looks as if it is probably (4, 7). Check:

$$\text{at } (4, 7) : P = 0.8 \times 4 + 7 = 10.2;$$

$$\text{at } (5, 6) : P = 0.8 \times 5 + 6 = 10;$$

$$\text{at } (6, 5) : P = 0.8 \times 6 + 5 = 9.8.$$

This gives the optimal solution that 4 fruit cakes and 7 iced cakes should be made to yield an income of £10.20.

- (b) Changing the price of the iced cakes changes the function which is to be maximized to:

$$P = 0.8x + 1.5y$$

It does not change the feasible region. Two  $P$ -lines corresponding to the new  $P$ -function are shown in Figure 21.

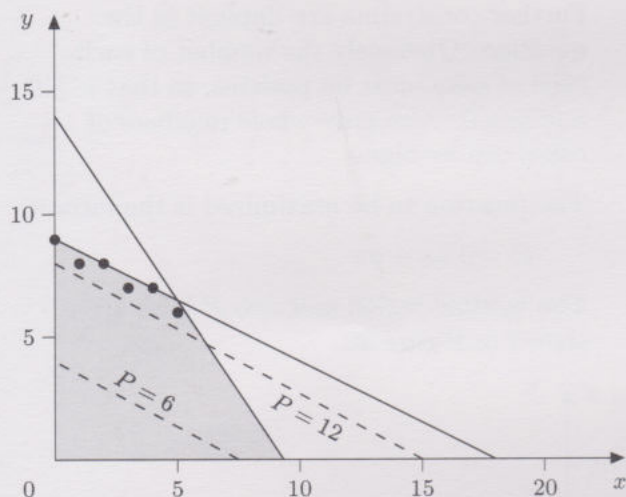


Figure 21

Again, the maximum value occurs at the vertex (5, 6.5), which is non-integer. The candidates for the best integer-valued point are (0, 9), (1, 8), (2, 8), (3, 7), (4, 7) and (5, 6). The points are marked on Figure 21. Clearly (2, 8) is better than (1, 8) and (4, 7) is better than (3, 7). Checking the other four points:

$$\text{at } (0, 9) : P = 0.8 \times 0 + 1.5 \times 9 = 13.5;$$

$$\text{at } (2, 8) : P = 0.8 \times 2 + 1.5 \times 8 = 13.6;$$

$$\text{at } (4, 7) : P = 0.8 \times 4 + 1.5 \times 7 = 13.7;$$

$$\text{at } (5, 6) : P = 0.8 \times 5 + 1.5 \times 6 = 13.$$

So, to maximize your income, you should make 4 fruit cakes and 7 iced cakes to yield £13.70. So, increasing the price of iced cakes to £1.50 does not change the optimum numbers of cakes that you should bake.

- (c) The new cost function is:

$$P = 0.8x + 1.7y$$

Figure 22 shows two  $P$ -lines corresponding to this new cost function.

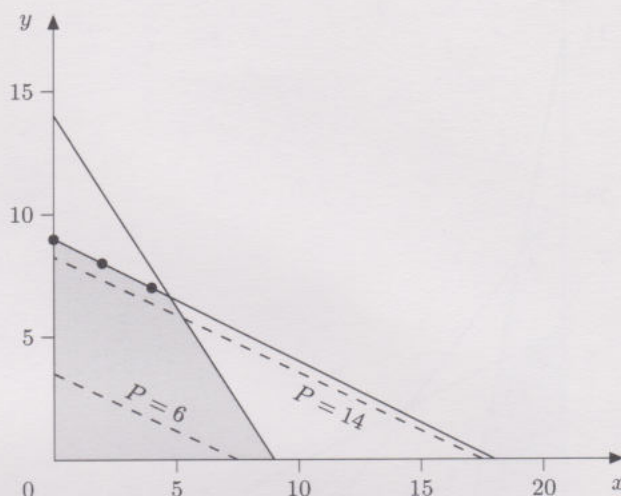


Figure 22

This time it looks like the maximum value occurs at a different vertex, at the point (0, 9) where  $x + 2y = 10$  cuts the  $y$ -axis. This point is integer-valued. But, to check, because the  $P$ -lines are close to being parallel to  $x + 2y = 10$ , it is worth evaluating  $P$  at the points (0, 9), (2, 8) and (4, 7) shown on the figure.

$$\text{at } (0, 9) : P = 0.8 \times 0 + 1.7 \times 9 = 15.3;$$

$$\text{at } (2, 8) : P = 0.8 \times 2 + 1.7 \times 8 = 15.2;$$

$$\text{at } (4, 7) : P = 0.8 \times 4 + 1.7 \times 7 = 15.1.$$

The change in price to £1.70 for an iced cake has now made a difference to the optimum number of cakes. It is now more advantageous, providing an income of £15.30, to make 9 iced cakes and no fruit cakes, as long as this is acceptable to the club.

### 13

- (a) The restrictions given in the question translate to the following constraints:

$$8x + 16y \geq 75 \quad (\text{protein})$$

$$46x + 6y \geq 250 \quad (\text{carbohydrate})$$

$$2x + 8y \geq 70 \quad (\text{fat})$$

Also implicit in the question are the restrictions that the amount of bread and cheese in the diet are positive, so that  $x \geq 0$  and  $y \geq 0$ . The daily calorie intake, which must be minimized, is given by:

$$P = 235x + 166y$$



Figure 23 shows the feasible region.

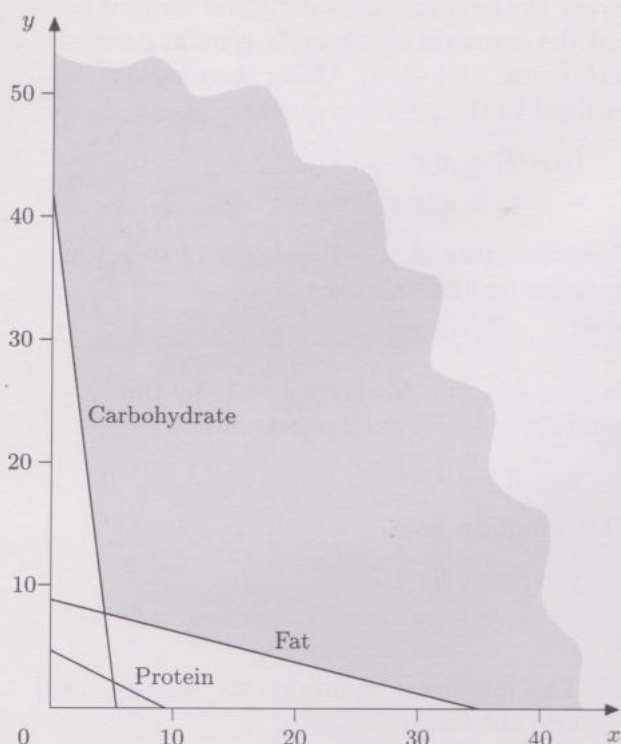


Figure 23

- (b) The line representing the minimum protein requirement does not intersect the feasible region. The conclusion that can be drawn from this is that the protein requirement is automatically satisfied if the other requirements are met.
- (c) Two  $P$ -lines for decreasing values of  $P$  are shown in Figure 24.

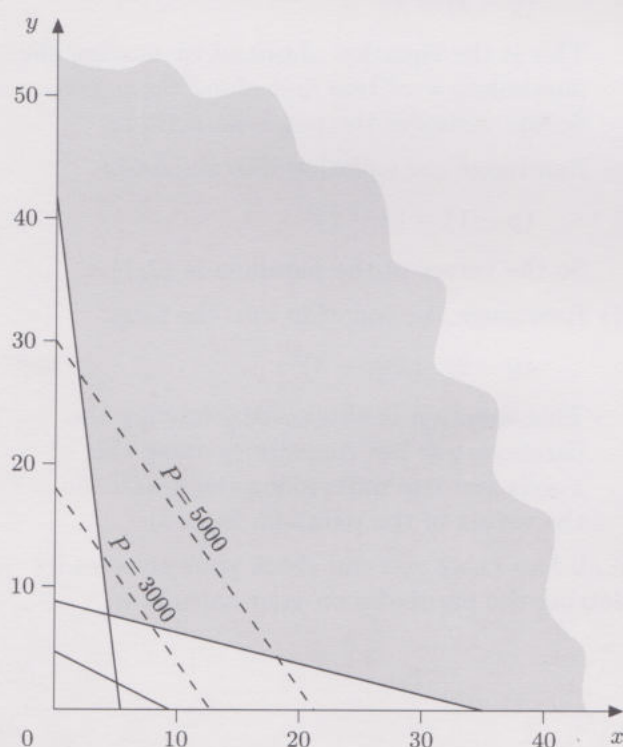


Figure 24

From the figure, the point that minimizes the calorie intake is at the intersection of the lines representing the minimum carbohydrate and fat requirements—that is, at approximately  $(4.5, 7.5)$ . Interpreting this solution in terms of the original problem gives the optimum diet of eating 450 g of bread and 750 g of cheese per day.

- (d) The coordinates of the optimal solution can be worked out more accurately algebraically, by solving simultaneously the equations of the lines corresponding to the minimum carbohydrate requirement and the minimum fat requirement.

$$46x + 6y = 250 \quad (9)$$

$$2x + 8y = 70 \quad (10)$$

Rearranging (10) gives

$$x = 35 - 4y \quad (11)$$

which can be substituted into (9):

$$46(35 - 4y) + 6y = 250$$

$$1610 - 184y + 6y = 250$$

$$-178y = -1360$$

$$y = 7.64$$

(to two decimal places)

From (11),

$$x = 35 - 4 \times 7.64 = 4.44$$

(to two decimal places)

Interpreting this solution in terms of the original problem gives the optimum diet of eating 444 g of bread and 764 g of cheese per day.

- (e) The daily calorie intake for this diet is  $235 \times 4.44 + 166 \times 7.64 = 2310$  calories per day (rounded to the nearest ten).

## Unit 11

### 1

- (a) To move a parabola  $k$  units along the  $x$ -axis, substitute  $(x - k)$  for every occurrence of  $x$  in the equation. Similarly, to move a parabola  $l$  units along the  $y$ -axis, substitute  $(y - l)$  for every occurrence of  $y$  in the equation.

- (i) To move the parabola with equation  $y = x^2$  by 4 units along the  $y$ -axis, substitute  $(y - 4)$  for  $y$  in the equation:

$$(y - 4) = x^2$$

$$y = x^2 + 4$$

This is the equation of a parabola with vertex at  $(0, 4)$ .

(Other parabolas are also possible—for example,  $y = 2x^2 + 4$ ,  $y = -x^2 + 4$ , or anything of the form  $y = ax^2 + 4$ .)

(ii) To move the parabola with equation  $y = x^2$  by 4 units along the  $x$ -axis, substitute  $(x - 4)$  for  $x$  in the equation:

$$\begin{aligned}y &= (x - 4)^2 \\y &= x^2 - 8x + 16\end{aligned}$$

This is the equation of a parabola with vertex at  $(4, 0)$ .

(Other parabolas are also possible—for example,  $y = 2(x - 4)^2$ ,  $y = -(x - 4)^2$ , or anything of the form  $y = a(x - 4)^2$ .)

(iii) To move the parabola with equation  $y = x^2$  so that its vertex is  $(1, -2)$ , substitute  $(x - 1)$  for  $x$  and  $(y - (-2)) = (y + 2)$  for  $y$  in the equation:

$$\begin{aligned}(y + 2) &= (x - 1)^2 \\y &= (x - 1)^2 - 2 \\y &= x^2 - 2x - 1\end{aligned}$$

This is the equation of a parabola with vertex at  $(1, -2)$ .

(Other possibilities are also possible—for example,  $y = 2(x - 1)^2 - 2$ ,  $y = -(x - 1)^2 - 2$ , or anything of the form  $y = a(x - 1)^2 - 2$ .)

- (b) This part of the question asks for the equations of *two* parabolas with vertex at  $(6, -4)$ . So you could move two equations of parabolas with vertex at the origin (for example,  $y = x^2$  and  $y = 2x^2$ ) to  $(6, -4)$ . Moving the general equation of a parabola with the vertex at the origin,  $y = ax^2$ , gives:

$$\begin{aligned}(y - (-4)) &= a(x - 6)^2 \\(y + 4) &= a(x^2 - 12x + 36) \\(y + 4) &= ax^2 - 12ax + 36a \\y &= ax^2 - 12ax + 36a - 4\end{aligned}$$

Now to obtain the equations of two parabolas with vertex at  $(6, -4)$  you can pick any two non-zero values for  $a$ .

$$\begin{aligned}a = 1 \text{ gives } y &= x^2 - 12x + 32 \\a = 2 \text{ gives } y &= 2x^2 - 24x + 68\end{aligned}$$

(You may have chosen other values for  $a$ .)

You could use your calculator to check your answers to this question, by drawing the graphs of your chosen parabolas.

**2** Begin with the most general parabola with vertex at the origin ( $y = ax^2$ ) and move it to find the equation of the most general parabola with vertex at  $(-1, 2)$ . Using the method outlined in the previous question gives:

$$\begin{aligned}(y - 2) &= a(x - (-1))^2 \\y &= a(x + 1)^2 + 2\end{aligned}$$

Now substitute in  $x = 1$  and  $y = 14$  to get an equation for the unknown  $a$ :

$$14 = a \times 2^2 + 2 = 4a + 2$$

So  $4a = 12$ , and this gives  $a = 3$ . So the equation of the desired parabola is:

$$y = 3(x + 1)^2 + 2$$

This simplifies to:

$$y = 3x^2 + 6x + 5$$

### 3

- (a) This quadratic equation can be rearranged into the form:

$$(y - 1) = (x + 1)^2$$

This equation is obtained by moving the parabola  $y = x^2$  by  $-1$  along the  $x$ -axis and  $1$  along the  $y$ -axis. Hence the vertex  $(0, 0)$  of the parabola  $y = x^2$  moves by  $-1$  along the  $x$ -axis and  $1$  along the  $y$ -axis to the point  $(-1, 1)$ . Hence the vertex of the given parabola is  $(-1, 1)$ .

- (b) Rearrange the equation into the form:

$$(y - 1) = x^2$$

This is the equation obtained by moving the parabola  $y = x^2$  one unit along the  $y$  axis. So the vertex of the parabola is  $(0, 1)$ .

- (c) Rearrange the equation into the form:

$$(y - 1) = (x - 2)^2$$

So the vertex of the parabola is  $(2, 1)$ .

- (d) Rearrange the equation into the form:

$$(y - 2) = 2(x - 3)^2$$

This equation is obtained by moving the parabola  $y = 2x^2$  three units along the  $x$ -axis and two units along the  $y$ -axis. So the vertex of the parabola is  $(3, 2)$ .

In all four cases you can check your answers by plotting the parabolas on your calculator.



4

- (a) On the calculator a linear fit looks quite reasonable given the amount of random variation that is present in the data. In the absence of any other information, a linear model is a reasonable choice.

- (b) The regression line is

$$y = 1080x - 323$$

(with parameters rounded to the nearest whole number). This gives a prediction for the weight at month 20 of

$$1080 \times 20 - 323 = 21\,300 \text{ grams}$$

(to the nearest hundred grams).

(Remember, when using a fitted model on your calculator, there is a shortcut you can use. Instead of reading the model parameters off the screen and typing them in again to make predictions, you can enter the regression equation as a function—see the *Calculator Book* for details.)

- (c) The best fit parabola is

$$y = 43x^2 + 563x + 795$$

(with parameters rounded to the nearest whole number.) This gives a prediction for the weight in month 20 of

$$43 \times 20^2 + 563 \times 20 + 795 = 29\,300 \text{ grams}$$

(to the nearest hundred grams).

- (d) The difference between the predictions is quite large. Comparing these predictions with the measurement in month 20 would be a very good test of the surface-area theory of coral growth.

Even though the linear fit looks good and has a high correlation coefficient ( $r = 0.987$ ), the predicted weight could be incorrect if the surface-area growth theory is a good model.

This shows the dangers inherent in fitting data without a theoretical model of the underlying behaviour.

5

- (a) The height of the cliff is the position of the stone initially (at time  $t = 0$ ). Reading this off the graph gives the answer 100 m.
- (b) A stationary stone does not change height, so any time period over which the stone is stationary should be represented by a horizontal line. Initially the graph is horizontal so this must represent the time between starting the watch and throwing the stone. The length of this initial line represents 5 seconds.

- (c) The cliff is 100 m high, so trace from 100 m on the height axis across to the curve and then down to the time axis. This means that at time  $t = 10$  seconds the stone returns to the height of the cliff edge.
- (d) By inspection, the graph is at the highest point approximately 7.5 seconds after the watch was started and reached a height of 135 m.
- (e) Collecting the data from the previous parts of the question, you want the equation of the parabola through the points (5, 100), (10, 100) and (7.5, 135). The calculator gives the quadratic regression curve passing through these points to be:
- $$y = -5.6x^2 + 84x - 180$$
- (f) The stone stops falling and comes to rest at the point where the curve becomes a horizontal line again, which is at about 22 metres above sea level. This must be the height of the bottom of the cliff.
- (g) By inspecting the graph the stone reaches the bottom of the cliff 12 seconds after the watch was started. Using your calculator to find the gradient of the parabola  $y = -5.6x^2 + 84x - 180$  when  $x = 12$  gives a gradient of  $-50.4 \text{ ms}^{-1}$ , which is interpreted as a speed of  $50.4 \text{ ms}^{-1}$  vertically downwards.

6

- (a) Using your calculator, simply input three points on the parabola from the graph: for example,  $(-1000, 0)$ ,  $(0, 0.015)$ ,  $(1000, 0)$ . The calculator then gives the following best fit curve:

$$y = -1.5 \times 10^{-8}x^2 + 0.015$$

Alternatively you can calculate the equation algebraically. The equation of a parabola with axis parallel to the  $y$ -axis and having vertex at  $(k, l)$  is:

$$y = a(x - k)^2 + l$$

The vertex of the given parabola is at  $(0, 0.015)$ , so the equation is of the form:

$$y = ax^2 + 0.015$$

Then one of the other known points (for example,  $(1000, 0)$ ) is substituted into this equation to find  $a$ :

$$0 = a \times 1000^2 + 0.015$$

So  $a = -1.5 \times 10^{-8}$  and the parabola has the equation:

$$y = -1.5 \times 10^{-8}x^2 + 0.015$$



- (b) Calculating the population density when  $x = 200$  gives:

$$y = -1.5 \times 10^{-8} \times 200^2 + 0.015 = 0.0144$$

So the predicted population density 200 m from the centre of the village is 0.0144 people per square metre.

7

- (a) Figure 25 shows how the population density is likely to vary along  $AB$ . The curve could be a parabola. It has been assumed that the population density is zero outside the village boundary.

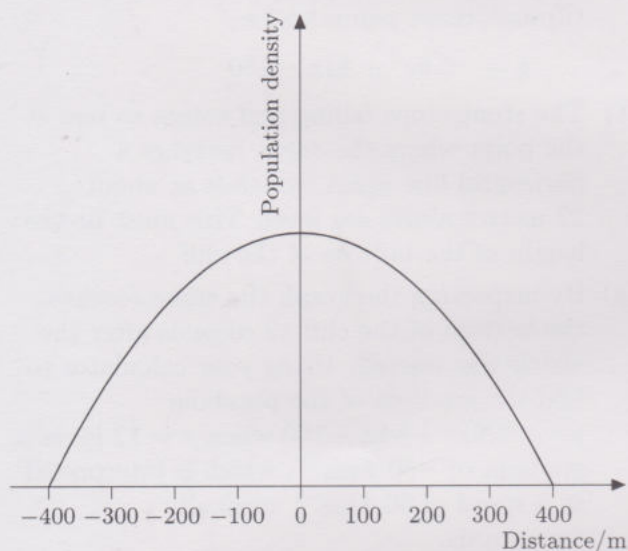


Figure 25

- (b) The following assumptions are needed.
- ◇ The population density is zero outside the village (as stated above).
  - ◇ The population density varies only with distance at right angles to the road and not at all along lines parallel to the road (this makes  $AB$  typical of all other such lines in the village).
  - ◇ The population density varies quadratically with the distance from the main road along lines like  $AB$ , and is symmetrical about the main road.
  - ◇ Regions where no one lives (gardens, roads, open areas, and so on) can be allowed for by 'smoothing out' the variation of population density with distance.

The variables will be the population density,  $y$  people  $\text{m}^{-2}$ , and the distance from the main road,  $x$  m. (Distances to the north will be positive, to the south negative.) The model will be valid only for  $-400 \leq x \leq 400$ .

Using your calculator and quadratic regression, with the data points  $(-400, 0)$ ,  $(0, 0.01)$  and  $(400, 0)$ , gives the equation:

$$y = -6.25 \times 10^{-8}x^2 + 0.01$$

Alternatively, algebraically the equation relating  $y$  and  $x$  will be of the form:

$$y = a(x - k)^2 + l$$

As the peak population density is along the road and is 0.01 people  $\text{m}^{-2}$ , the vertex of the parabola is at  $(0, 0.01)$  and so  $k = 0$  and  $l = 0.01$ . The equation therefore becomes:

$$y = ax^2 + 0.01$$

The population is 0 at a distance of 400 m from the road, so  $y = 0$  at  $x = 400$ . Using this gives:

$$0 = a(400)^2 + 0.01$$

Therefore:

$$a = \frac{-0.01}{(400)^2} = -6.25 \times 10^{-8}$$

Hence the equation is:

$$y = -6.25 \times 10^{-8}x^2 + 0.01$$

This equation can be used to predict  $y$  at  $x = 200$ :

$$\begin{aligned} y &= -6.25 \times 10^{-8} \times (200)^2 + 0.01 \\ &= 0.0075 \end{aligned}$$

So the population density is about 0.0075 people  $\text{m}^{-2}$  at 200 m from the road.

## Unit 12

1

- (a) The population increases by tripling so the first four members of the sequence are 5, 15, 45, 135. The formula for the  $n$ th generation can most easily be seen by writing down the first few terms in a way that makes the pattern evident.

| Generation | Population         |
|------------|--------------------|
| 1          | 5                  |
| 2          | $5 \times 3$       |
| 3          | $5 \times 3^2$     |
| 4          | $5 \times 3^3$     |
| $\vdots$   | $\vdots$           |
| $n$        | $5 \times 3^{n-1}$ |

So the population in the 20th generation is  $5 \times 3^{19} = 5811\,307\,335$ .



- (b) Adding 1% is equivalent to multiplying by 1.01. So the mass of the fungus (rounded to one decimal place) after the first four days is as shown below.

| Generation | Mass (g)             |        |
|------------|----------------------|--------|
| 1          | 23.1                 | = 23.1 |
| 2          | $23.1 \times 1.01$   | = 23.3 |
| 3          | $23.1 \times 1.01^2$ | = 23.6 |
| 4          | $23.1 \times 1.01^3$ | = 23.8 |

This shows the pattern for the general formula. The mass of the fungus in the  $n$ th day is  $23.1 \times 1.01^{n-1}$  grams. This means that in the 20th day the mass of the fungus is 27.9 grams.

- (c) Subtracting 10% is equivalent to multiplying by 0.9. So the balance after the first four months is shown below.

| Generation | Balance (£)         |        |
|------------|---------------------|--------|
| 1          | 1000                | = 1000 |
| 2          | $1000 \times 0.9$   | = 900  |
| 3          | $1000 \times 0.9^2$ | = 810  |
| 4          | $1000 \times 0.9^3$ | = 729  |

This shows that the balance after  $n$  months is  $\pounds 1000 \times 0.9^{n-1}$ , and so the balance after 20 months is  $\pounds 1000 \times 0.9^{19} = \pounds 135.09$  (to the nearest penny).

## 2

- (a) Using the answer to Question 1, the sum of the first four generations is  
 $5 + 15 + 45 + 135 = 200$ .

The general formula is:

$$S(n) = a \left( \frac{b^n - 1}{b - 1} \right)$$

In this case  $a = 5$  and  $b = 3$ , so the formula for the sum of the first  $n$  generations for this model is:

$$S(n) = 5 \left( \frac{3^n - 1}{3 - 1} \right) = \frac{5}{2} (3^n - 1)$$

So the sum of the first 20 generations is:

$$S(20) = \frac{5}{2} (3^{20} - 1) = 8716\,961\,000$$

- (b) The sum of the first four generations is:  
 $23.1 + 23.3 + 23.6 + 23.8 = 93.8$  grams.

For this model  $a = 23.1$  and  $b = 1.01$ , so the formula for the sum of the first  $n$  generations is:

$$\begin{aligned} S(n) &= 23.1 \left( \frac{1.01^n - 1}{1.01 - 1} \right) \\ &= 2310(1.01^n - 1) \end{aligned}$$

So the sum of the first 20 generations is:

$$S(20) = 2310(1.01^{20} - 1) = 508.6 \text{ grams}$$

- (c) The sum of the first four generations is  
 $1000 + 900 + 810 + 729 = 3439$ .

In this case  $a = 1000$  and  $b = 0.9$ , so the formula for the sum of the first  $n$  generations is:

$$\begin{aligned} S(n) &= 1000 \left( \frac{0.9^n - 1}{0.9 - 1} \right) \\ &= -10\,000(0.9^n - 1) \end{aligned}$$

Note that both  $-10\,000$  and  $(0.9^n - 1)$  in the above formula are negative, and so it is more usual to multiply both by  $-1$  to make them both positive. So the more usual form is:

$$S(n) = 10\,000(1 - 0.9^n)$$

So the sum of the first 20 generations is:

$$S(20) = 10\,000(1 - 0.9^{20}) = \pounds 8784.23$$

## 3

- (a) This equation can be solved by taking  $\log_{10}$  of both sides:

$$\log_{10}(20) = \log_{10}(10^x)$$

$$\log_{10}(20) = x$$

$$x = 1.30 \quad (\text{to two decimal places})$$

- (b) Simplify the equation to get  $\log_{10}(x)$  on its own:

$$0.25 = 1 / \log_{10} x$$

$$\log_{10} x = 1 / 0.25$$

$$\log_{10} x = 4$$

Now raise 10 to the power of both sides:

$$10^{\log_{10} x} = 10^4$$

$$x = 10^4 = 10\,000$$

- (c) To solve this equation, use the fact that  $\log_{10}(x^3) = 3 \times \log_{10}(x)$  and simplify:

$$2 \log_{10}(x^3) = 6$$

$$6 \log_{10} x = 6$$

$$\log_{10} x = 1$$

$$x = 10^1 = 10$$

- (d) Take  $\log_{10}$  of both sides of the equation:

$$2^x = 100$$

$$\log_{10}(2^x) = \log_{10} 100$$

$$x \times \log_{10} 2 = 2$$

$$x = \frac{2}{\log_{10} 2}$$

$$x = 6.64 \quad (\text{to two decimal places})$$

4

- (a) The 110th month is the first time the balance dips below £0.01, since:

$$1000 \times 0.9^{109} = 0.0103 \text{ (to 4 d.p.)}$$

$$1000 \times 0.9^{110} = 0.0093 \text{ (to 4 d.p.)}$$

So the last penny is spent in about the 110th month.

- (b) The equation to be solved is:

$$0.01 = 1000 \times 0.9^{n-1}$$

This can be solved by taking logarithms of both sides and simplifying:

$$\log_{10}(0.01) = \log_{10}(1000 \times 0.9^{n-1})$$

$$\log_{10}(0.01) = \log_{10}(1000) + \log_{10}(0.9^{n-1})$$

$$\log_{10}(0.01) = \log_{10}(1000)$$

$$+ (n-1) \times \log_{10}(0.9)$$

$$-2 = 3 + (n-1) \times \log_{10}(0.9)$$

$$\frac{-5}{\log_{10}(0.9)} = n-1$$

$$109.27 \simeq n-1$$

$$n \simeq 110$$

5

- (a) To find the population doubling time you need to solve the equation:

$$2 \times 209 = 209 \times 3^x$$

This simplifies to:

$$2 = 3^x$$

Taking logarithms of both sides:

$$\log_{10} 2 = \log_{10}(3^x)$$

$$\log_{10} 2 = x \times \log_{10} 3$$

$$\frac{\log_{10} 2}{\log_{10} 3} = x$$

$$x = 0.63 \text{ (to two decimal places)}$$

So the population doubling time is 0.63 time units.

- (b) To find the population doubling time you need to solve the equation:

$$2 \times 0.43 = 0.43 \times 1.2^x$$

This simplifies to:

$$2 = 1.2^x$$

Taking logarithms of both sides:

$$\log_{10} 2 = \log_{10}(1.2^x)$$

$$\log_{10} 2 = x \times \log_{10}(1.2)$$

$$\frac{\log_{10} 2}{\log_{10}(1.2)} = x$$

$$x = 3.80 \text{ (to two decimal places)}$$

So the population doubling time is 3.8 time units.

- (c) Since the population is decreasing, the population will never double!

But if you did not realize this and applied the same method as the previous parts of the question, then the equation to solve is:

$$2 \times 65 = 65 \times (0.9)^x$$

This simplifies to:

$$2 = 0.9^x$$

Taking logarithms of both sides:

$$\log_{10} 2 = \log_{10}(0.9^x)$$

$$\log_{10} 2 = x \times \log_{10}(0.9)$$

$$\frac{\log_{10} 2}{\log_{10}(0.9)} = x$$

$$x = -6.58 \text{ (to two decimal places)}$$

The answer is negative, so this means that for the population to double you have to go back 6.58 time units—that is, 6.58 time units ago the population was twice what it is now. (So the half-life of the population is 6.58 time units.)

6

- (a) The exponential model is expressed by an equation of the form:

$$y = a \times b^x$$

When  $x = 0$  in the above equation then  $y = a$ . The initial population in this case is  $y = 76$ , so  $a = 76$ . Substituting this into the above equation gives:

$$y = 76 \times b^x$$

The population doubling time is 2 minutes, so  $y = 2 \times 76 = 152$  when  $x = 2$ .

Substituting these values into the equation gives:

$$152 = 76 \times b^2$$

So  $b^2 = 2$  and  $b = \sqrt{2} \simeq 1.414$ , and the exponential model equation is:

$$y = 76 \times (1.414)^x$$

- (b) Using the same method as before, the initial population (10) is equal to  $a$ . The doubling time is 6 minutes, so  $y = 20$  when  $x = 6$ . This is substituted in to the exponential model equation to determine  $b$ :

$$20 = 10 \times b^6$$

So  $b^6 = 2$  and  $b = 2^{1/6} \simeq 1.122$ , and the exponential model equation is:

$$y = 10 \times (1.122)^x$$



- (c) The initial population gives  $a = 1000$ . The population has halved in 23 minutes, so  $y = 500$  when  $x = 23$ . Substituting into the equation gives:

$$500 = 1000 \times b^{23}$$

So  $b = (0.5)^{1/23} \simeq 0.9703$ , and the exponential model equation is:

$$y = 1000 \times (0.9703)^x$$

7

- (a) If there are  $N$  atoms of uranium 239 at the start of an experiment, then after 23.5 minutes there will  $N/2$  atoms. So after another 23.5 minutes there will be half of  $N/2$  atoms, namely  $N/4$  atoms. This is exactly the time period that you are looking for, the time for the number of atoms to reduce by a factor of four. So the 'quarter-life' of the isotope is 47 minutes.
- (b) The easiest way to solve this part of the question is to find the equation of the decay first. Let the decay constant be  $b$ , then the equation describing the number of uranium atoms is:

$$y = Nb^x$$

After 23.5 minutes the number of atoms has halved, so:

$$N/2 = Nb^{23.5}$$

$$0.5 = b^{23.5}$$

$$b = (0.5)^{1/23.5}$$

$$b \simeq 0.971$$

Now the 'third-life' can be calculated; it is the time for the number of uranium atoms to decrease to one third. So, substituting  $N/3$  for  $y$  in the equation

$$y = N \times 0.971^x$$

gives:

$$N/3 = N \times 0.971^x$$

$$\frac{1}{3} = 0.971^x$$

This equation can be solved by taking  $\log_{10}$  of both sides:

$$\log_{10} \frac{1}{3} = \log_{10}(0.971^x)$$

$$\log_{10} \frac{1}{3} = x \times \log_{10}(0.971)$$

$$x = \frac{\log_{10} \frac{1}{3}}{\log_{10}(0.971)}$$

$$x \simeq 37.3$$

So the 'third-life' of uranium 239 is about 37.3 minutes. (As a check, this time should be somewhere between the half-life and the 'quarter-life', which it is.)

8

- (a) Let  $p$  be the bird population at time  $t$ . Using the calculator (and rounding to three significant figures to match the data), gives the best fit exponential model as:

$$p = 555 \times (0.880)^t$$

(Since  $r = -0.993$  this model fits the data well.)

The model gives a prediction for the population in the year 2000 of  $555 \times (0.88)^{15} = 82$  birds (to the nearest whole number).

- (b) Using a linear model for the data produces:

$$p = 514 - 40x$$

(Since  $r = -0.993$  this model is also a good fit.)

The model gives a prediction for the population in the year 2000 of  $514 - 40 \times 15 = -86$  birds (to the nearest whole number).

- (c) The models predict significantly different numbers of birds. The number of birds predicted by the linear model is negative, which means that this model predicts that the species will be extinct by the year 2000 (it predicts that the extinction will occur about year 12 or 13). However, both models predict that the population will be in dire trouble.

9

- (a) When the course calculator evaluates the expression

$$1.000\,000\,000\,001 - 1$$

the correct answer is displayed (that is,  $1 \times 10^{-12}$ ). But for the next expression,

$$1.000\,000\,000\,0001 - 1$$

the calculator displays the wrong answer 0. This is not a fault with the calculator but is a direct result of the finite precision of the calculator. The same behaviour can be observed at some point for any calculator (or computer).

So the smallest value of  $b$  for which the course calculator gives the correct answer is  $1.000\,000\,000\,001$ .

- (b) To find when the given exponential model gives a  $y$ -value bigger than 10, you need to solve the following:

$$10 = 1.000\,000\,000\,001^x$$



This can be solved by taking logarithms of both sides of the equation:

$$\begin{aligned}\log_{10} 10 &= \log_{10}(1.000\,000\,000\,001^x) \\ 1 &= x \times \log_{10}(1.000\,000\,000\,001) \\ x &\simeq 2.30 \times 10^{12}\end{aligned}$$

So the exponential is bigger than 10 for any  $x$  bigger than about  $2.30 \times 10^{12}$ .

- (c) Set the calculator to display a table of  $1.000\,000\,000\,001^x$  and  $1000x^2 + 1000$  values, starting at  $x = 1 \times 10^{13}$  and in steps of  $1 \times 10^{13}$ , and you will see how rapidly the exponential function is changing in this range. The exponential function overtakes the polynomial between  $7 \times 10^{13}$  and  $8 \times 10^{13}$ .

(To solve this problem exactly you would have to solve the equation

$$1000x^2 + 1000 = 1.000\,000\,000\,001^x$$

for which the only known methods of solution are various computer search algorithms.)

## Unit 13

### 1

- (a) (i) The variables are number of pages and time to read them. Call them  $p$  and  $t$  (in hours).

(ii) Assume a constant reading rate of 20 pages per hour.

$$(iii) \quad t = \frac{1}{20}p$$

The constant of proportionality is the time to read one page—that is,  $\frac{1}{20}$  (of an hour).

- (b) (i)  $t = \frac{1}{20} \times 115 = 5.75$  hours or about five and three-quarter hours.

(ii)  $t = \frac{1}{20} \times 46 = 2.3$  hours or about two hours twenty minutes.

(iii)  $t = \frac{1}{20} \times 670 = 33.5$  hours or about thirty-three and a half hours.

Note these answers have been rounded to the nearest five minutes, as the model is unlikely to give the prediction more accurately than this!

- 2 The depth  $d$  is inversely proportional to the square of the diameter of the tin. So the depth would be scaled by a factor of  $(15/20)^2 = 0.5625$ , and so the depth of the food in the dish would almost be halved.

- 3 If the height of the cakes is to be the same then the volume of the cake is proportional to the area of the base of the cake tin. For a square tin of side  $x$  inches, the area is  $x^2$  square inches. For a round tin of diameter  $d$  inches, the radius is  $d/2$  inches and so the area is  $\pi(d/2)^2$  square inches. Equating these two areas gives:

$$x^2 = \pi(d/2)^2 \simeq 0.785 d^2$$

This relationship can be simplified by taking the square root of both sides to get:

$$x = 0.886 d$$

For an 8-inch round cake tin,  $d = 8$ ; and so  $x = 0.886 \times 8 \simeq 7$ . So a 7-inch square tin is equivalent to an 8-inch round tin.

### 4

- (a) To obtain a simple relationship between cooking times and microwave power you must assume that the amount the food is cooked depends only on the total amount of microwave energy input. This means that:

$$\begin{aligned}\text{amount food is cooked} &\propto \text{energy} \\ &= \text{power rating} \\ &\quad \times \text{cooking time}\end{aligned}$$

So the cooking time is inversely proportional to the power rating of the oven.

(This relationship works well for a given item of food but is not very good for translating cooking times between different items. For different food items other factors come into play: for example, the cooking time also depends on the shape of the food and the proportion of the microwave power absorbed by the food.)

- (b) The model is probably valid only over a small range of microwave power, say from about 100 watts to 1000 watts, because it relies on the food responding linearly to the amount of microwave energy input. For small power ratings the approximation will fail because the heat will dissipate by conduction out of the food and the food will not reach a high enough temperature to cook. For very large power ratings the foodstuff's ability to absorb microwaves will be saturated.
- (c) If  $A$  is the cooking time for the 500-watt oven and  $B$  is the cooking time for the 650-watt oven, then from part (a):

$$500A = 650B$$



Therefore:

$$A = \frac{650}{500}B = \frac{13}{10}B \quad \text{and} \quad B = \frac{500}{650}A = \frac{10}{13}A$$

These formulas can be used to complete the table as follows.

| Food item            | Cooking time (minutes) |                |
|----------------------|------------------------|----------------|
|                      | 500-watt oven          | 650-watt oven  |
| One corn-on-the-cob  | $6\frac{1}{2}$         | 5              |
| 225 g oven chips     | 9                      | 7              |
| Whole kipper         | 8                      | 6              |
| 4 × 225 g cod steaks | $15\frac{1}{2}$        | 12             |
| 2 kg beef            | 64                     | 49             |
| 2 × 100 g pork chops | 6                      | $4\frac{1}{2}$ |

5

- (a) Since you are told that both screens are the standard shape, the television screens must be mathematically similar figures and one is just a scaled version of another. So the 21-inch screen is scaled up by  $24/21$  to get the 24-inch screen. So the area is scaled up by this factor squared:  $(24/21)^2 \simeq 1.31$ . So the 24-inch television would be perceived to be 1.31 times, or 31%, bigger than the 21-inch one.

Alternatively, start by drawing a diagram of the two television screen, as in Figure 26.

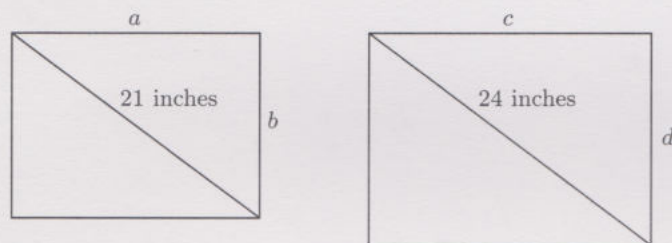


Figure 26

All the dimensions in the figure must be in the same ratio as the diagonals, namely  $21:24$ . So  $c = (24/21)a$  and  $d = (24/21)b$ . Now calculate the area of the bigger screen.

$$\begin{aligned} \text{area} &= c \times d = \frac{24a}{21} \times \frac{24b}{21} \\ &= \left(\frac{24}{21}\right)^2 \times a \times b \\ &\simeq 1.31 \times a \times b \end{aligned}$$

So the screen area of the larger television is 1.31 times the screen area of the smaller television. So people would perceive the larger television to be 1.31 times, or 31%, bigger.

- (b) The argument in the previous part applies. This gives the following equation:

$$\begin{aligned} &\text{area of large television} \\ &= \left(\frac{x}{21}\right)^2 \times \text{area of 21-inch television} \end{aligned}$$

So the perceived increase in size will be:

$$\text{increase} = \left(\frac{x}{21}\right)^2$$

It is this quadratic increase in perceived size which makes televisions with only a slightly larger diagonal *seem* much larger.

6

- (a) The surface area of the sweets is proportional to  $r^2$  and the volume to  $r^3$ . So as  $r$  goes up by a factor of 3 the surface coating will go up by a factor of  $3^2 = 9$  and the interior will go up by a factor of  $3^3 = 27$ .
- (b) The weight of the sweets will scale in exactly the same proportion as the volume. So one giant sweet will weigh 27 times as much as one small sweet, or to put it another way one small sweet will weigh  $1/27$  as much as a giant sweet. So 600 small sweets will weigh the same as  $600/27 \simeq 22$  giant sweets. So 22 giant sweets would need to be put in a packet.
- (c) Each giant sweet needs 9 times as much coating, but there are now 22 sweets in a packet instead of 600. So for a packet of giant sweets the coating ingredients would have to be scaled by  $9 \times 22/600 = 0.33$ .
- For the interior of the sweets, each giant sweet needs 27 times more ingredients, but there are now 22 sweets in a packet instead of 600. So for a packet the interior ingredients would have to be scaled by  $27 \times 22/600 = 0.99$ .
- (d) The scaling for 100 packets would be the same as the scaling for one packet. So the answer remains the same: the interior ingredients would be roughly unchanged (scaling factor 0.99) but the coating ingredients would need to be scaled down to about a third (scaling factor 0.33).

7

- (a) Let  $t$  be the time taken to dig the hole. The volume of the hole to be dug is  $x \times x \times x/2 = x^3/2$ . Assume that the amount of time taken to dig a hole is directly proportional to the volume of soil that has to be moved, so

$$t = k \times x^3/2$$

where  $k$  is a constant. To determine  $k$ , use the data from the test hole. When  $x = 1$  m then  $t = 2$  hours. Substitute this into the equation:

$$2 = k \times 1^3/2$$

This gives  $k = 4$ . The formula for the time taken is then as follows:

$$t = 2 \times x^3$$

- (b) For this case  $x = 5$ , so:

$$t = 2 \times 5^3 = 250 \text{ hours}$$

So the time taken to dig the whole swimming pool is 250 hours. (Based on this large estimate for the time, it might be more sensible to pay to hire an excavator!)

8

- (a)  $I$  is the energy per unit area per second; so, to find the total energy per second, multiply by the area:

$$\begin{aligned} \text{total energy} &= \text{intensity} \times \text{area} \\ &= I \times 4\pi r^2 \end{aligned}$$

- (b) The total energy per second calculated in the previous part of the question must equal the energy emitted by the star per second. Therefore:

$$E = I \times 4\pi r^2$$

This gives the relationship between intensity and distance as:

$$I = \frac{E}{4\pi} \times r^{-2}$$

This is an inverse square proportionality relationship and is known in physics as an *inverse square law*. Such laws often occur in physics for exactly the same reasons as above (that is, something is being distributed equally in all directions).

(To use the above formula, the amount of light energy  $E$  emitted by the star would have to be known. One way of determining this is to use a relationship between  $E$  and the colour of the light emitted. The colour of the light emitted by a star is often used to help describe the star: for example, red giant, brown dwarf, and so on.)

- 9 The correct pairings are 1e, 2d, 3c, 4b, 5a and 6f.





